Comput. Methods Funct. Theory (2013) 13:47–63
DOI 10.1007/s40315-012-0003-x

Normality of Meromorphic Functions and Uniformly Discrete Exceptional Sets

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Received: 11 May 2012 / Revised: 10 September 2012 / Accepted: 8 December 2012 /
Published online: 23 January 2013
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Abstract Let $k \in \mathbb{N}$ and $h(\not\equiv 0)$ be a function holomorphic on D. Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least $k+3$. Suppose that the sets $\{E_f\}_{f\in\mathcal{F}}$ are locally uniformly discrete in D, where $E_f = \{z \in D : f(z) = 0\} \cup \{z \in D : f^{(k)}(z) = h(z)\}$. Suppose additionally that at the common zeros of $f \in \mathcal{F}$ and h, the multiplicities m_f for f and m_h for h satisfy $m_f \geq m_h + k + 1$ for $k > 1$ and $m_f \geq 2m_h + 3$ for $k = 1$. Then, $\mathcal{F}$ is normal in D. The number $k+3$ can be replaced by $k+2$ if the set E_f is independent of f, or in other words, for each pair of functions f and g in $\mathcal{F}$, f and g share the value 0, and $f^{(k)}$ and $g^{(k)}$ share the function h. Examples are also given to show that the conditions are necessary and sharp.

Keywords Meromorphic function · Normal family · Shared values · Locally uniformly discrete sets

Mathematics Subject Classification (2000) Primary 30D45

1 Introduction and Main Results

A family $\mathcal{F}$ of meromorphic functions defined in a plane domain $D \subset \mathbb{C}$ is said to be normal in D, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally uniformly in D to a meromorphic function or ∞. See [6,11,15].

Communicated by Lawrence Zalcman.

Research supported by NNSF of China, Grant No. 11171045.

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The Gu's normality criterion [5] which was conjectured by Hayman [6] says that a family $\mathcal{F}$ of functions meromorphic on D is normal if $f \neq 0$ and $f^{(k)} \neq 1$ for each $f \in \mathcal{F}$. The following generalization of Gu's theorem was proved by Yang [14].

Theorem 1 *Let $\mathcal{F}$ be a family of meromorphic functions on D, $k \in \mathbb{N}$ and $h(\not\equiv 0)$ be a holomorphic function on D. If for every $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq h$ on D, then $\mathcal{F}$ is normal on D.*

In recent years, following Schwick [12], many normality criteria concerning shared values or functions have been proved. We say that two functions f and g share a value or a function ϕ if the two equations $f(z) = \phi(z)$ and $g(z) = \phi(z)$ have the same solutions (ignoring multiplicity). Here, we want to generalize the following result of Fang and Zalcman [4] by replacing the constant 1 by a function.

Theorem 2 *Let k be a positive integer and let $\mathcal{F}$ a family of meromorphic functions on D, all of whose zeros have multiplicity at least $k+2$, such that for each pair of functions f and g in $\mathcal{F}$, f and g share the value 0, and $f^{(k)}$ and $g^{(k)}$ share the value 1. Then, the family $\mathcal{F}$ is normal.*

In general, the constant 1 cannot be replaced by a function. For example, the family $\{f_n\}$, where $f_n(z) = nz^{k+2}$, is not normal at 0. However, each pair of functions f_n and f_m share the value 0, and $f_n^{(k)}$ and $f_m^{(k)}$ share the function z^2.

So we need some additional conditions. We prove the following generalization of Theorem 2.

Theorem 3 *Let $k \in \mathbb{N}$ and $h(\not\equiv 0)$ be a function holomorphic on D. Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least $k+2$, such that for each pair of functions f and g in $\mathcal{F}$, f and g share the value 0, and $f^{(k)}$ and $g^{(k)}$ share the function h. Suppose additionally that at each common zero of f and h for every $f \in \mathcal{F}$, the multiplicities m_f for f and m_h for h satisfy $m_f \geq m_h + k + 1$ for $k > 1$ and $m_f \geq 2m_h + 3$ for $k = 1$. Then, $\mathcal{F}$ is normal in D.*

The above example shows that the assumption $m_f \geq m_h + k + 1$ is necessary. The condition $m_f \geq 2m_h + 3$ for $k = 1$ is also necessary and sharp as showed by the following example.

Example 1 Let α be a positive integer and $h(z) = z^\alpha$. Let for $n \in \mathbb{N}$,

$$f_n(z) = \frac{z^{2\alpha+2}}{(\alpha+1)[z^{\alpha+1} - 1/n]}.$$

Then,

$$f_n'(z) - h(z) = \frac{z^{3\alpha+2} - \frac{2}{n} z^{2\alpha+1}}{\left(z^{\alpha+1} - \frac{1}{n}\right)^2} - z^\alpha = -\frac{\frac{1}{n^2} z^\alpha}{\left(z^{\alpha+1} - \frac{1}{n}\right)^2}.$$

Thus, each pair of functions f_n and f_m share the value 0, and f_n' and f_m' share the function $h(z)$. However, the family $\{f_n\}$ is not normal at 0.

Let us look at some more aspects. By fixing a function $f_0 \in \mathcal{F}$ and letting $E = \{z \in D : f_0(z) = 0\} \cup \{z \in D : f_0^{(k)}(z) = h(z)\}$, the shared condition of Theorem 3 is equivalent to saying that there exists a fixed set $E \subset D$, which is independent of $f \in \mathcal{F}$, such that $f \neq 0$ and $f^{(k)} \neq h$ on $D \setminus E$ for every $f \in \mathcal{F}$. There are two cases for the set E. One which is trivial is that $E = D$. Then, every $f \in \mathcal{F}$ satisfies $f \equiv 0$ or $f^{(k)} \equiv h$, and hence the normality of $\mathcal{F}$ can be easily dealt with. The non-trivial case is that the exceptional set E is (locally) discrete in D. We focus on the non-trivial case and consider here that the exceptional set E is dependent on f. To state our result, we require the following definition.

Definition 1 The sets $\{E_\lambda\}_{\lambda \in \Lambda}$ are said to be locally uniformly discrete in D, if for each point $z_0 \in D$, there exists $\delta > 0$ such that every E_λ has at most one point lying in the disk $\Delta(z_0, \delta) = \{z : |z - z_0| < \delta\}$.

For example, the sets $\{E_n\}_{n \in \mathbb{N}}$, where $E_n = \{\frac{m-1}{m} + \frac{1}{n} : m \in \mathbb{N}\}$, are locally uniformly discrete in the unit disk $\Delta(0, 1)$, but not in the domains that contain the point 1.

Theorem 4 *Let $k \in \mathbb{N}$ and $h (\not\equiv 0)$ be a function holomorphic on D. Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least $k+2$ or $k+3$ and h has simple zeros. Suppose that the sets $\{E_f\}_{f \in \mathcal{F}}$, where $E_f = \{z \in D : f(z) = 0\} \cup \{z \in D : f^{(k)}(z) = h(z)\}$, are locally uniformly discrete in D. Suppose additionally that at the common zeros of $f \in \mathcal{F}$ and h, the multiplicities m_f for f and m_h for h satisfy $m_f \geq m_h + k + 1$ for $k > 1$ and $m_f \geq 2m_h + 3$ for $k = 1$. Then, $\mathcal{F}$ is normal in D.*

The following example shows that it is necessary (and sharp) to assume that the multiplicity of zeros of $f \in \mathcal{F}$ is at least $k+3$ when h has simple zeros.

Example 2 Let $h(z) = z$ and

$$f_n(z) = \frac{(z - 1/n)^{k+2}}{(k+1)![z - (k+2)/n]}, \quad n \in \mathbb{N}.$$

Then, $E_{f_n} = \{1/n\}$, so that $\{E_{f_n}\}$ are locally uniformly discrete in $\mathbb{C}$. In fact, we have

$$f_n(z) = \frac{\left[\left(z - \frac{k+2}{n}\right) + \frac{k+1}{n}\right]^{k+2}}{(k+1)!\left(z - \frac{k+2}{n}\right)} = \frac{1}{(k+1)!}\left[\left(z - \frac{k+2}{n}\right)^{k+1} + \frac{(k+1)(k+2)}{n}\left(z - \frac{k+2}{n}\right)^{k} + P(z) + \frac{\left(\frac{k+1}{n}\right)^{k+2}}{\left(z - \frac{k+2}{n}\right)}\right],$$

where P is a polynomial of degree $< k$, so that

$$f_n^{(k)}(z) = z + \frac{(-1)^k \left(\frac{k+1}{n}\right)^{k+2}}{(k+1)\left(z - \frac{k+2}{n}\right)^{k+1}} \neq h(z).$$

However, $\{f_n\}$ is not normal at 0, as $f_n(1/n) = 0$ while $f_n((k+2)/n) = \infty$. Throughout in this paper, we denote by $\mathbb{C}$ the complex plane, by $\mathbb{C}^*$ the punctured complex plane $\mathbb{C} \setminus \{0\}$, by $\Delta(z_0, r)$ the open disk $\{z : |z - z_0| < r\}$, and by $\Delta^\circ(z_0, r)$ the punctured disk $\Delta(z_0, r) \setminus \{z_0\} = \{z : 0 < |z - z_0| < r\}$, where $z_0 \in \mathbb{C}$ and $r > 0$.

2 Auxiliary Results

To prove our results, we require some preliminary results.

Lemma 5 *[3] Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least k. Then, if $\mathcal{F}$ is not normal at z_0, there exist, for each $-1 < \alpha < k$, points $z_n \in D$ with $z_n \to z_0$, functions $f_n \in \mathcal{F}$ and positive numbers $\rho_n \to 0$ such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges locally uniformly with respect to the spherical metric in $\mathbb{C}$ to a non-constant meromorphic function g of finite order, all of whose zeros have multiplicity at least k.*

The original form ($\alpha = 0$) of this rescaling lemma is due to Zalcman [16], while the case $-1 < \alpha < 1$ was proved by Pang [8,9]. The present form is due to Chen and Gu [3]. This lemma also holds for $\alpha = k$ [10] under an additional condition.

Lemma 6 *[6,7] Let k be a positive integer, f be a transcendental meromorphic function and $P(\not\equiv 0)$ be a polynomial. Then, either f or $f^{(k)} - P$ has infinitely many zeros. If f is a non-constant rational function, then either f or $f^{(k)} - 1$ has at least one zero.*

Lemma 7 *[13] Let k be a positive integer and let f be a non-constant rational function such that $f^{(k)} \neq 1$ on $\mathbb{C}$. Then, either f is a polynomial of degree at most k or $f(z) = z^k/k! + P_k(z) + a(z-b)^{-n}$, where $a(\neq 0)$, $b \in \mathbb{C}$ and $n \in \mathbb{N}$ are constants, and P_k is a polynomial of degree less than k. Furthermore, f has a zero whose multiplicity is at most $k+1$.*

Lemma 8 *[2, Lemma 4] Let k be a positive integer and f be a non-constant rational function. If $f(z) \neq 0$ for $z \in \mathbb{C}$, then $f^{(k)} - 1$ has at least $k+1$ distinct zeros on $\mathbb{C}$.*

Lemma 9 *Let k be a positive integer and f be a non-constant rational function. If $f(z)[f^{(k)}(z) - 1] \neq 0$ for $z \in \mathbb{C} \setminus \{z_0\}$, where $z_0 \in \mathbb{C}$, then z_0 is a zero of f with multiplicity at most $k+1$.*

Proof We claim that $f(z_0) = 0$. For otherwise, we would have $f(z) \neq 0$ for $z \in \mathbb{C}$ by the condition, and hence, by Lemma 8, $f^{(k)} - 1$ has at least $k+1 \geq 2$ distinct zeros, which contradicts that $f^{(k)}(z) \neq 1$ for $z \neq z_0$.

We now assume that the zero z_0 of f has multiplicity at least $k+2$. Then, $f^{(k)}(z_0) = 0$. Hence by the condition, we have $f(z) \neq 0$ for $z \in \mathbb{C} \setminus \{z_0\}$, and $f^{(k)}(z) \neq 1$ for $z \in \mathbb{C}$. Thus by Lemma 7, f has a zero whose multiplicity is at most $k+1$. Since $f(z) \neq 0$ for $z \in \mathbb{C} \setminus \{z_0\}$, this zero coincides with z_0, which contradicts the assumption that the zero z_0 of f has multiplicity at least $k+2$.

Thus, z_0 is a zero of f with multiplicity at most $k+1$. □

Lemma 10 *Let k, m be positive integers and f be a non-constant rational function. If $f^{(k)}(z) \neq z^m$ for $z \in \mathbb{C}$ and if $f(z) \neq 0$ for $z \neq z_0$, where $z_0 \in \mathbb{C}$, then $m = 1$, $z_0 \neq 0$ and either $f(z) = (z - z_0)^{k+1}/(k+1)!$ or*

$$f(z) = \frac{1}{(k+1)!} \cdot \frac{(z - z_0)^{k+2}}{z - (k+2)z_0}.$$

Proof Consider first the case that f is a non-constant polynomial. Then by $f(z) \neq 0$ for $z \neq z_0$, $f(z) = C_1(z - z_0)^l$ for some constant $C_1 \neq 0$ and $l \in \mathbb{N}$; and by $f^{(k)}(z) \neq z^m$, $f^{(k)}(z) = z^m + C_2$ for some constant $C_2 \neq 0$. Thus, $C_1[(z - z_0)^l]^{(k)} = z^m + C_2$. It can be easily seen that $z_0 \neq 0$, $m = 1$, $l = k + 1$ and $C_1 = 1/(k+1)!$. Hence $f(z) = (z - z_0)^{k+1}/(k+1)!$.

Now we assume that f is a non-polynomial rational function. By $f^{(k)}(z) \neq z^m$ for $z \in \mathbb{C}$, we have

$$\left[f(z) - \frac{m!}{(m+k)!} z^{m+k} + \frac{1}{k!} z^k \right]^{(k)} = f^{(k)}(z) - z^m + 1 \neq 1.$$

Thus by Lemma 7,

$$f(z) = \frac{m!}{(m+k)!} z^{m+k} + P_k(z) + \frac{a}{(z-b)^n}, \tag{1}$$

where P_k, a, b, n are stated as in Lemma 7. Since $f(z) \neq 0$ for $z \neq z_0$, we also get $f(z) = C(z - z_0)^l (z - b)^{-n}$, for some constant $C \neq 0$ and integer $l \geq 0$. This, combined with (1), yields that

$$\left[\frac{m!}{(m+k)!} z^{m+k} + P_k(z) \right] (z-b)^n = C(z - z_0)^l - a. \tag{2}$$

Comparing the degree and the coefficient of the leading term of (2) yields that $l = m + k + n$ and $C = \frac{m!}{(m+k)!}$. Further, since each zero of the right hand side of (2) is simple, we see that $n = 1$. Thus, $l = m + k + 1$ and we can deduce from (2) that

$$\left[\frac{m!}{(m+k)!} z^{m+k} + P_k(z) \right] (z-b) = \frac{m!}{(m+k)!} (z - z_0)^{m+k+1} - a. \tag{3}$$

Now by comparing the coefficients of the term z^{m+k}, we get $b = (m + k + 1)z_0$.

We claim that $z_0 \neq 0$. In fact, if $z_0 = 0$, then $b = 0$, and hence $a = 0$ by taking $z = 0$ in (3). This is a contradiction.

Next by comparing the coefficients of the term z^{m+k-1}, we see that $m = 1$. Thus, f has the second desired form. □

Lemma 11 *Let k, m be positive integers and f be a rational function. If $f(z) \neq 0$ for $z \in \mathbb{C}$, and $f^{(k)}(z) \neq z^m$ for $z \neq z_0$, where $z_0 \in \mathbb{C}$, then f is a constant.*

Proof If f is a polynomial, then by $f(z) \neq 0$, f must be constant. Now suppose that f is a non-polynomial rational function. Then by Lemma 10, $f^{(k)}(z) - z^m$ must have at least one zero. Hence by the condition, $f^{(k)}(z_0) = z_0^m$. Thus by $f(z) \neq 0$ for $z \in \mathbb{C}$ and $f^{(k)}(z) \neq z^m$ for $z \neq z_0$, we can write

$$f(z) = C_1 \prod_{i=1}^{n} (z - z_i)^{-p_i}, \ \ f^{(k)}(z) = z^m + C_2 (z - z_0)^l \prod_{i=1}^{n} (z - z_i)^{-p_i - k}, \tag{4}$$

where C_1, C_2 are non-zero constants, l, n, p_i are positive integers, and $z_i, 0 \leq i \leq n$ are distinct complex numbers. By the expression of f in (4), we have

$$f^{(k)}(z) = P(z) \prod_{i=1}^{n} (z - z_i)^{-p_i - k}, \tag{5}$$

where P is polynomial of degree $(n-1)k$. Thus by the two expressions of $f^{(k)}$ in (4) and (5),

$$z^m \prod_{i=1}^{n} (z - z_i)^{p_i + k} + C_2 (z - z_0)^l = P(z). \tag{6}$$

Since

$$\deg(P) = (n-1)k < m + \sum_{i=1}^{n} (p_i + k) = \deg\left(z^m \prod_{i=1}^{n} (z - z_i)^{p_i + k} \right),$$

by (6), we have

$$l = m + \sum_{i=1}^{n} (p_i + k) = m + kn + \sum_{i=1}^{n} p_i \tag{7}$$

and $C_2 = -1$. By letting $z = 1/t$ in (6), we get

$$\prod_{i=1}^{n} (1 - z_i t)^{p_i + k} = (1 - z_0 t)^l + t^l P(1/t) = (1 - z_0 t)^l \left[1 + O(t^{m+k+\sum_{i=1}^{n} p_i}) \right] \tag{8}$$

as t goes to 0. Thus by taking the logarithmic derivatives,

$$\sum_{i=1}^{n} \frac{(p_i + k) z_i}{1 - z_i t} - \frac{l z_0}{1 - z_0 t} = O(t^{m+k-1+\sum_{i=1}^{n} p_i}). \tag{9}$$

It follows that

$$\sum_{i=1}^{n}(p_i+k)z_i^j - lz_0^j = 0 \ \text{ for } \ 1 \le j \le m+k-1+\sum_{i=1}^{n} p_i. \tag{10}$$

Since $m + k - 1 + \sum_{i=1}^{n} p_i \ge n + 1$, it follows that the system of linear equations

$$\sum_{i=0}^{n} z_i^j x_i = 0, \quad 1 \le j \le n+1 \tag{11}$$

has a non-zero solution $(x_0, x_1, \ldots, x_n) = (-l, p_1 + k, \ldots, p_n + k)$. This is impossible, as all z_i are distinct. □

Lemma 12 *Let k, m be positive integers, and let f be a non-constant rational function. If $f(z)[f^{(k)}(z) - z^m] \ne 0$ for $z \ne 0$, and the multiplicity is at least $m + k + 1$ when 0 is a zero of f, then $k = 1$ and 0 is a zero of f with exact multiplicity $2m + 2$.*

Proof First, we show that f cannot be a polynomial. Suppose not, then by $f(z) \ne 0$ for $z \ne 0$, $f(z) = Cz^s$ for some constant $C \ne 0$ and integer $s \in \mathbb{N}$. Further, by the condition, $s \ge m+k+1$. Thus, $f^{(k)}(z)-z^m = Az^{s-k} - z^m = Az^m(z^{s-k-m} - 1/A)$, where $A \ne 0$ is a constant. This contradicts that $f^{(k)}(z) - z^m \ne 0$ for $z \ne 0$.

Thus, f is a non-polynomial rational function. By Lemma 11, f has at least one zero, and hence by $f(z) \ne 0$ for $z \ne 0$, we must have $f(0) = 0$. Thus, we can write

$$f(z) = C_1 z^l \prod_{i=1}^{n}(z - z_i)^{-p_i}, \tag{12}$$

where $C_1 \ne 0$ is constant, $z_i \in \mathbb{C}$, $1 \le i \le n$ are distinct and non-zero, and $n, l, p_i \in \mathbb{N}$ with $l \ge m + k + 1$ (by the condition). Thus, 0 is a zero of $f^{(k)}$ with multiplicity $l-k \ge m+1$, and hence 0 is a zero of $f^{(k)}(z)-z^m$ with exact multiplicity m. Hence, since $f^{(k)}(z) - z^m \ne 0$ for $z \ne 0$, we have

$$f^{(k)}(z) = z^m + C_2 z^m \prod_{i=1}^{n}(z - z_i)^{-p_i-k}, \tag{13}$$

for some constant $C_2 \ne 0$. However, by (12), one can obtain by induction that

$$f^{(k)}(z) = C_1 z^{l-k} P(z) \prod_{i=1}^{n}(z - z_i)^{-p_i-k}, \tag{14}$$

where

$$P(z) = \prod_{j=0}^{k-1}\left(l - j - \sum_{i=1}^{n} p_i\right) z^{nk} + \cdots \ (\not\equiv 0) \tag{15}$$

is a polynomial of degree at most nk. Thus by the two expressions of $f^{(k)}$ in (13) and (14),

$$z^m \prod_{i=1}^{n}(z-z_i)^{p_i+k} + C_2 z^m = C_1 z^{l-k} P(z),$$

and hence

$$\prod_{i=1}^{n}(z-z_i)^{p_i+k} = -C_2 + C_1 z^{l-k-m} P(z). \tag{16}$$

Then, comparing the degrees of the both sides of (16) shows that

$$\sum_{i=1}^{n}(p_i+k) = l-k-m+\deg(P). \tag{17}$$

Since $\deg(P) \le nk$, it follows from (17) that $l \ge m+k+\sum_{i=1}^{n} p_i$. Thus by (15), $\deg(P) = nk$ and then by (17),

$$l = m+k+\sum_{i=1}^{n} p_i. \tag{18}$$

By (16), we also have

$$C_2 = -\prod_{i=1}^{n}(-z_i)^{p_i+k}. \tag{19}$$

Now by taking the logarithmic derivatives of (16), we have

$$\sum_{i=1}^{n}\frac{p_i+k}{z-z_i} = O(z^{l-k-m-1}) \ \text{ as } \ z \to 0. \tag{20}$$

Thus,

$$\sum_{i=1}^{n}\frac{p_i+k}{(z_i)^j} = 0 \ \text{ for } \ 1 \le j \le l-k-m-1. \tag{21}$$

It follows that the system of linear equations

$$\sum_{i=1}^{n}\frac{x_i}{(z_i)^j} = 0, \quad 1 \le j \le l-k-m-1 \tag{22}$$

has a non-zero solution $(x_1, \ldots, x_n) = (p_1 + k, \ldots, p_n + k)$. Since z_i are non-zero and distinct, we get $l - k - m - 1 < n$, and hence by (18), $\sum_{i=1}^{n} p_i < n + 1$. It follows that all $p_i = 1$. Thus, $l = m + k + n$ and by (21),

$$\sum_{i=1}^{n} \frac{1}{(z_i)^j} = 0, \quad j = 1, \ldots, n - 1. \tag{23}$$

It follows from the well-known Newton's formula that $\prod_{i=1}^{n}(z - z_i) = z^n - r$, where $r \neq 0$ is a constant. Hence by (19)

$$C_2 = -\prod_{i=1}^{n}(-z_i)^{p_i+k} = -(-r)^{k+1}. \tag{24}$$

Thus by (12) and (13),

$$f(z) = C_1 z^{k+m+n}(z^n - r)^{-1} = C_1 z^{k+m}(1 - rz^{-n})^{-1}, \tag{25}$$

$$\begin{aligned} f^{(k)}(z) &= z^m - (-r)^{k+1} z^m (z^n - r)^{-k-1} \\ &= z^m \left[1 - (-rz^{-n})^{k+1}(1 - rz^{-n})^{-k-1}\right]. \end{aligned} \tag{26}$$

By (25), we have

$$f(z) = C_1 z^{k+m} \sum_{s=0}^{\infty} (rz^{-n})^s = C_1 \sum_{s=0}^{\infty} r^s z^{k+m-sn}$$

for z satisfying $|rz^{-n}| < 1$, and hence

$$\begin{aligned} f^{(k)}(z) &= C_1 \sum_{s=0}^{\infty} r^s \prod_{j=1}^{k}(j + m - sn) z^{m-sn} \\ &= C_1 z^m \sum_{s=0}^{\infty} \prod_{j=1}^{k}(j + m - sn)(rz^{-n})^s. \end{aligned} \tag{27}$$

Thus by (26) and (27) with writing $w = rz^{-n}$,

$$1 - (-w)^{k+1}(1 - w)^{-k-1} = C_1 \sum_{s=0}^{\infty} \prod_{j=1}^{k}(j + m - sn) w^s \quad \text{for } |w| < 1. \tag{28}$$

Now comparing the coefficients of (28) yields that $\prod_{j=1}^{k}(j + m - sn) = 0$ for $1 \leq s \leq k$. Thus, $sn - m \in \{1, 2, \ldots, k\}$ for each $1 \leq s \leq k$. In particular, $1 \leq n - m \leq k$ and $1 \leq kn - m \leq k$. It follows that $k = 1$ and $n = m + 1$, and hence 0 is a zero of f with exact multiplicity $l = m + k + n = 2m + 2$. $\square$

3 Proof of Theorem 4

Let $z_0 \in D$ be a point and $\{f_n\} \subset \mathcal{F}$ be a sequence. We have to prove that $\{f_n\}$ has a subsequence which is normal at z_0. We may assume that $z_0 = 0$. Since $\{E_{f_n}\}$ are locally uniformly discrete in D, there exists $\delta_0 > 0$ such that $E_{f_n} \cap \Delta(0, \delta_0)$ contains at most one point z_{f_n}. That is to say, we have $f_n \neq 0$ and $f_n^{(k)} \neq h$ on $\Delta(0, \delta_0) \setminus \{z_{f_n}\}$ for all f_n. The following considerations for $\{f_n\}$ are understood to always hold with respect to the disk $\Delta(0, \delta_0)$.

Case 1 There exists $0 < \delta < \delta_0$ such that $|z_{f_n}| \geq \delta$ for all f_n (with n sufficiently large). Then, we have $f_n \neq 0$ and $f_n^{(k)} \neq h$ on $\Delta(0, \delta)$ for all f_n. Thus by Theorem 1, $\{f_n\}$ is normal on $\Delta(0, \delta)$, and hence at $z_0 = 0$.

Case 2 There exists a subsequence of $\{z_{f_n}\}$, which we continue to call $\{z_{f_n}\}$, such that $z_{f_n} \to z_0 = 0$. Then by Theorem 1, $\{f_n\}$ is normal on $\Delta^\circ(0, \delta_0)$.

Suppose that $\{f_n\}$ has no subsequence which is normal at 0. Next we consider two cases according to whether $h(0)$ is 0 or not.

Case 3 For the case $h(0) \neq 0$, we may say that $h(0) = 1$. Since $\{f_n\}$ is not normal at 0, by Lemma 5, there exist a subsequence of $\{f_n\}$ which we continue to call $\{f_n\}$, a sequence of points $z_n \to 0$ and a sequence of positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) := \rho_n^{-k} f_n(z_n + \rho_n \zeta) \to g(\zeta) \tag{29}$$

spherically locally uniformly on $\mathbb{C}$, where g is non-constant and meromorphic on $\mathbb{C}$.

Claim 1 $g^{(k)}(\zeta) \not\equiv 1$. *In fact, if $g^{(k)}(\zeta) \equiv 1$, then g is a polynomial with exact degree k, so that g has a zero $\zeta_0 \in \mathbb{C}$ with multiplicity at most k, and hence by applying Hurwitz's theorem to (29), g_n (for sufficiently large n) has a zero $\zeta_n \to \zeta_0$ with multiplicity at most k. It follows that f_n has a zero $z_n + \rho_n \zeta_n \to 0$ with multiplicity at most k. This contradicts the assumption that all zeros of f_n have multiplicity at least $k + 2$.*

Claim 2 *g has at most one zero, and if it has, then the multiplicity is at least $k + 2$.*

The latter assertion follows from an argument similar to that in Claim 1. We now prove the former. Suppose that g has at least two distinct zeros $\zeta_1, \zeta_2 \in \mathbb{C}$. Then by applying Hurwitz's theorem to (29), g_n (for sufficiently large n) has two distinct zeros $\zeta_n^{(1)}, \zeta_n^{(2)}$ tending to ζ_1, ζ_2 respectively, and hence f_n has two distinct zeros $z_n + \rho_n \zeta_n^{(1)}$ and $z_n + \rho_n \zeta_n^{(2)}$, both tending to 0. This contradicts that $f_n(z) \neq 0$ for $z \neq z_{f_n}$.

Claim 3 *$g^{(k)} - 1$ has at most one zero.*

By Claim 1 and the fact that

$$f_n^{(k)}(z_n + \rho_n \zeta) - h(z_n + \rho_n \zeta) = g_n^{(k)}(\zeta) - h(z_n + \rho_n \zeta) \to g^{(k)}(\zeta) - 1 (\not\equiv 0) \tag{30}$$

locally uniformly on $\mathbb{C} \setminus g^{-1}(\infty)$, an argument similar to that in Claim 2 yields this claim.

Claim 4 *Either $g \neq 0$ or $g^{(k)} \neq 1$.*

Suppose not, say $g(\zeta_0^{(1)}) = 0$ and $g^{(k)}(\zeta_0^{(2)}) = 1$. Since $\zeta_0^{(1)}$ is a zero of g with multiplicity $\geq k+2$, we have $g^{(k)}(\zeta_0^{(1)}) = 0$, and hence $\zeta_0^{(1)} \neq \zeta_0^{(2)}$. By applying Hurwitz's theorem to (29) and (30), there exist points $\zeta_n^{(i)} \to \zeta_0^{(i)}$ such that $g_n(\zeta_n^{(1)}) = 0$ and $g_n^{(k)}(\zeta_n^{(2)}) - h(z_n + \rho_n\zeta_n^{(2)}) = 0$, and hence $f_n(z_n + \rho_n\zeta_n^{(1)}) = 0$ and $f_n^{(k)}(z_n + \rho_n\zeta_n^{(2)}) - h(z_n + \rho_n\zeta_n^{(2)}) = 0$. Since $f_n(z) \neq 0$ and $f_n^{(k)}(z) \neq h(z)$ for $z \neq z_{f_n}$, it follows that $z_n + \rho_n\zeta_n^{(1)} = z_n + \rho_n\zeta_n^{(2)} (= z_{f_n})$ so that $\zeta_n^{(1)} = \zeta_n^{(2)}$, and hence $\zeta_0^{(1)} = \zeta_0^{(2)}$. This is a contradiction.

Thus by Claims 2–4, $g(g^{(k)} - 1)$ has at most one zero, and hence by Lemma 6, g is a rational function. Further by Lemma 9, g has a zero with multiplicity at most $k+1$. This contradicts Claim 2 which says that the zero of g has multiplicity at least $k+2$.

Case 4 Next we consider the case that $h(0) = 0$. Then $h(z) = z^m\phi(z)$, where $m \in \mathbb{N}$ and ϕ is holomorphic with $\phi(0) \neq 0$. We can say $\phi \neq 0$ in $\Delta(0, \delta_0)$ with the normalization $\phi(0) = 1$. Set for each n

$$F_n(z) = z^{-m} f_n(z). \tag{31}$$

Then $F_n(z) \neq 0$ for $z \neq z_{f_n}$, since $f_n(z) \neq 0$ for $z \neq z_{f_n}$.

We first show that $\{F_n\}$ has no subsequence which is normal at 0. Suppose not, say $\{F_n\}$ is normal at 0. Then $\{F_n\}$ has a subsequence, which we continue to call $\{F_n\}$, such that $\{F_n\}$ converges spherically locally uniformly to ψ which may be ∞ identically in some neighborhood $\Delta(0, \eta_0)$.

If $\psi(0) \neq \infty$, then for sufficiently large n, $f_n(z) \neq \infty$ and $|F_n(z)| \leq M$ in some closed domain $\overline{\Delta}(0, \eta)$ with $\eta < \eta_0$, where $M > 0$ is a constant. It follows that the functions f_n are holomorphic and satisfy $|f_n(z)| \leq M|z|^m$ on $\overline{\Delta}(0, \eta)$. By the Montel's theorem, $\{f_n\}$ is normal at 0, which contradicts our assumption that $\{f_n\}$ is not normal at 0.

If $\psi(0) = \infty$, then for sufficiently large n, $|F_n(z)| > 1$ in some closed domain $\overline{\Delta}(0, \eta)$ with $\eta < \eta_0$. We claim that $f_n(z_{f_n}) \neq 0$ for all (sufficiently large) n. In fact, if $f_n(z_{f_n}) = 0$, then as $F_n(z_{f_n}) \neq 0$, we see that $z_{f_n} = 0$, and so by the assumption on the multiplicities of common zeros of f_n and h, $z_{f_n} = 0$ is a zero of f_n with multiplicity at least $m + k + 1 > m$, and hence $F_n(0) = 0$, which contradicts that $|F_n(z)| > 1$. Thus, $f_n(z_{f_n}) \neq 0$. This, combined with the fact that $f_n(z) \neq 0$ for $z \neq z_{f_n}$, shows that the functions $1/f_n$ are holomorphic. By $|F_n(z)| > 1$, we have $|1/f_n(z)| < |z|^{-m}$ on $\overline{\Delta}(0, \eta)$. Now the maximum modulus principle implies that $|1/f_n(z)| \leq \eta^{-m}$ on $\overline{\Delta}(0, \eta)$ and hence $\{1/f_n\}$ is normal at 0 by Montel's theorem. This again contradicts our assumption that $\{f_n\}$ is not normal at 0.

Thus, $\{F_n\}$ has no subsequence which is normal at 0.

Claim 5 *If z_{f_n} is a zero of F_n, then the multiplicity is at least $k+1$, or $k+2$ if $z_{f_n} \neq 0$.* *In fact, suppose $F_n(z_{f_n}) = 0$, then $f_n(z_{f_n}) = 0$. Thus for the case $z_{f_n} \neq 0$, the claim is true by the assumption on the multiplicities of the zeros of f_n, while for the case $z_{f_n} = 0$, $z_{f_n} = 0$ is a common zero of f_n and h, and hence by the assumption on the multiplicities of the common zeros of f_n and h, 0 is a zero of f_n with multiplicity at least $k + m + 1$, and then the claim follows.*

Thus by Claim 5 and the fact that $F_n(z) \neq 0$ for $z \neq z_{f_n}$, Lemma 5 can be applied, and so there exist a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, a sequence of points $w_n \to 0$ and a sequence of positive numbers $\eta_n \to 0$ such that

$$\widehat{g}_n(\zeta) := \eta_n^{-k} F_n(w_n + \eta_n \zeta) = \eta_n^{-k}(w_n + \eta_n \zeta)^{-m} f_n(w_n + \eta_n \zeta) \to \widehat{g}(\zeta) \quad (32)$$

spherically locally uniformly on $\mathbb{C}$, where $\widehat{g}$ is non-constant and meromorphic on $\mathbb{C}$.

Claim 6 *$\widehat{g}$ has at most one zero.*

In fact, if $\widehat{g}$ has two distinct zeros $\zeta_1, \zeta_2 \in \mathbb{C}$, then applying Hurwitz's theorem to (32), there exist points $\zeta_{i,n} \to \zeta_i$ such that $\widehat{g}_n(\zeta_{i,n}) = 0$ and hence $F_n(w_n + \eta_n \zeta_{i,n}) = 0$. Since $F_n(z) \neq 0$ for $z \neq z_{f_n}$, we get $w_n + \eta_n \zeta_{1,n} = w_n + \eta_n \zeta_{2,n} (= z_{f_n})$. It follows that $\zeta_{1,n} = \zeta_{2,n}$, and hence $\zeta_1 = \zeta_2$. This is a contradiction.

Next we consider two subcases according to whether the sequence $\{w_n/\eta_n\}$ is bounded or unbounded.

Case 5 First assume that the sequence $\{w_n/\eta_n\}$ is unbounded. Then, there exists a subsequence, which we continue to call $\{w_n/\eta_n\}$, such that $w_n/\eta_n \to \infty$.

Claim 7 *If $\widehat{g}$ has a zero, then the multiplicity is at least $k+2$. And hence, $\widehat{g}^{(k)} \not\equiv 1$.*

In fact, if $\widehat{g}(\zeta_0) = 0$, then as above, there exist points $\zeta_n \to \zeta_0$ such that $F_n(w_n + \eta_n \zeta_n) = 0$, and hence $z_{f_n} = w_n + \eta_n \zeta_n$. As $w_n/\eta_n \to \infty$, we see that $z_{f_n} \neq 0$, and hence by Claim 5, z_{f_n} is a zero of F_n with multiplicity at least $k+2$. Thus, ζ_n is a zero of $\widehat{g}_n$ with multiplicity at least $k+2$. The claim then follows.

Claim 8 *$\widehat{g}^{(k)} - 1$ has at most one zero.*

This follows from a similar argument as above with the following fact that

$$\begin{aligned}
\frac{f_n^{(k)}(w_n + \eta_n \zeta)}{h(w_n + \eta_n \zeta)} &= \frac{[(\eta_n)^{-k} f_n(w_n + \eta_n \zeta)]^{(k)}}{h(w_n + \eta_n \zeta)} = \frac{[(w_n + \eta_n \zeta)^m \widehat{g}_n(\zeta)]^{(k)}}{h(w_n + \eta_n \zeta)} \\
&= \frac{\left[\left(\zeta + \frac{w_n}{\eta_n}\right)^m \widehat{g}_n(\zeta)\right]^{(k)}}{\left(\zeta + \frac{w_n}{\eta_n}\right)^m \phi(w_n + \eta_n \zeta)} = \frac{\sum_{i=0}^{k} \binom{k}{i} \left[\left(\zeta + \frac{w_n}{\eta_n}\right)^m\right]^{(k-i)} \widehat{g}_n^{(i)}(\zeta)}{\left(\zeta + \frac{w_n}{\eta_n}\right)^m \phi(w_n + \eta_n \zeta)} \\
&= \frac{1}{\phi(w_n + \eta_n \zeta)} \sum_{i=0}^{k} \frac{C_i \widehat{g}_n^{(i)}(\zeta)}{\left(\zeta + \frac{w_n}{\eta_n}\right)^{k-i}} \to \widehat{g}^{(k)}(\zeta) \qquad (33)
\end{aligned}$$

locally uniformly on $\mathbb{C} \setminus \widehat{g}^{-1}(\infty)$, where $C_i = m(m-1)\cdots(m-(k-i)+1)\binom{k}{i}$ are constants, and in particular, $C_k = 1$.

Claim 9 *Either $\widehat{g} \neq 0$ or $\widehat{g}^{(k)} \neq 1$.*

Combined with the fact that $F_n(z) \neq 0$ and $f_n^{(k)}(z) \neq h(z)$ for $z \neq z_{f_n}$, this can be shown by an argument similar to the proof of Claim 4 by applying Hurwitz's theorem to (32) and (33).

However, as in Case 2.1, the above Claims 6–9 lead to a contradiction.

Case 6 Now we consider the case that $\{w_n/\eta_n\}$ is bounded. Then, there is a subsequence, which we continue to call $\{w_n/\eta_n\}$, such that $w_n/\eta_n \to \alpha \in \mathbb{C}$. It follows from (32) that

$$\frac{f_n(\eta_n\zeta)}{\eta_n^{k+m}\zeta^m} = \widehat{g}_n\left(\zeta - \frac{w_n}{\eta_n}\right) \to \widehat{g}(\zeta - \alpha) \tag{34}$$

spherically locally uniformly on $\mathbb{C}$, and hence

$$G_n(\zeta) := \frac{f_n(\eta_n\zeta)}{\eta_n^{k+m}} = \zeta^m \cdot \frac{f_n(\eta_n\zeta)}{\eta_n^{k+m}\zeta^m} \to G(\zeta) := \zeta^m\widehat{g}(\zeta - \alpha) \tag{35}$$

spherically locally uniformly on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, or on $\mathbb{C}$ if $\widehat{g}(-\alpha) \neq \infty$. Obviously, G is meromorphic on $\mathbb{C}$ and $G \not\equiv 0$.

Claim 10 *G is non-constant. Suppose not. Then, $\widehat{g}(\zeta - \alpha) = A\zeta^{-m}$ for some non-zero constant A. Thus by (34), we see that*

$$G_n(\zeta) := \frac{f_n(\eta_n\zeta)}{\eta_n^{k+m}} \to A \tag{36}$$

locally uniformly on $\mathbb{C}$. Hence $G_n^{(k)}(\zeta) \to 0$, so that

$$\eta_n^{-m}[f_n^{(k)}(\eta_n\zeta) - h(\eta_n\zeta)] = G_n^{(k)}(\zeta) - \zeta^m\phi(\eta_n\zeta) \to -\zeta^m \tag{37}$$

locally uniformly on $\mathbb{C}$. Thus by applying Hurwitz's theorem to (37), there exist exactly m points $\zeta_n^{(j)} \to 0$, $j = 1, 2, \ldots, m$, such that $f_n^{(k)}(\eta_n\zeta_n^{(j)}) = h(\eta_n\zeta_n^{(j)})$. It follows from $f_n^{(k)}(z) \neq h(z)$ for $z \neq z_{f_n}$ that $z_{f_n} = \eta_n\zeta_n^{(j)}$ for all j. This shows that the m points $\zeta_n^{(j)}$ are coincide with $\zeta_n := z_{f_n}/\eta_n$, and z_{f_n} is a zero of $f_n^{(k)}(z) - h(z)$ with multiplicity m, and by (36), $f_n(z_{f_n}) \neq 0$. Thus, $f_n(z) \neq 0$ for $z \in \Delta(0, \delta_0)$.

Since $\{f_n\}$ is normal on $\Delta^\circ(0, \delta_0)$, but not normal at 0, it follows from $f_n \neq 0$ on $\Delta(0, \delta_0)$ that there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that $f_n \to 0$ and hence $f_n^{(k)} \to 0$ locally uniformly on $\Delta^\circ(0, \delta_0)$.

Thus by the argument principle, we see that

$$n\left(\frac{\delta_0}{2}, f_n^{(k)} - h\right) - n\left(\frac{\delta_0}{2}, \frac{1}{f_n^{(k)} - h}\right) \to n\left(\frac{\delta_0}{2}, h\right) - n\left(\frac{\delta_0}{2}, \frac{1}{h}\right) = -m, \tag{38}$$

where $n(r, f)$ is the number of poles of f in $\Delta(0, r)$, and $n(r, 1/f)$ is the number of zeros of f in $\Delta(0, r)$. Since both hand sides of (38) are integers, we see that for sufficiently large n,

$$n\left(\frac{\delta_0}{2}, f_n^{(k)} - h\right) = n\left(\frac{\delta_0}{2}, \frac{1}{f_n^{(k)} - h}\right) - m = 0. \tag{39}$$

It follows from (39) that f_n are holomorphic on $\Delta(0, \delta_0/2)$. Hence by $f_n \to 0$ locally uniformly on $\Delta^\circ(0, \delta_0)$, we get $f_n \to 0$ locally uniformly on $\Delta(0, \delta_0)$. This contradicts that $\{f_n\}$ is not normal at 0.

Claim 11 *G has at most one zero on $\mathbb{C}^*$, or on $\mathbb{C}$ if $\widehat{g}(-\alpha) \neq \infty$.*

This can be proved by applying Hurwitz's theorem to (35) with the fact that $f_n(z) \neq 0$ for $z \neq z_{f_n}$.

Claim 12 *If G has a zero on $\mathbb{C}^*$, then $z_{f_n} \neq 0$ and the multiplicity of the zero of G on $\mathbb{C}^*$ is at least $k+2$, or $k+3$ if $m = 1$.*

Suppose that $\zeta_0 \in \mathbb{C}^*$ is a zero of G. Then by applying Hurwitz's theorem to (35), G_n has a zero $\zeta_n \to \zeta_0$. Since $\zeta_0 \neq 0$, $\zeta_n \neq 0$ (for n large enough). It follows that $\eta_n \zeta_n \to 0$ is a non-zero zero of f_n. Since $f_n(z) \neq 0$ for $z \neq z_{f_n}$, we see that $z_{f_n} = \eta_n \zeta_n \neq 0$. Hence by the assumption, z_{f_n} is a zero of f_n with multiplicity at least $k+2$, or $k+3$ if $m = 1$. It follows that the multiplicity of the zero ζ_0 of G is at least $k+2$, or $k+3$ if $m = 1$.

Claim 13 *If $G(0) = 0$, then $f_n(0) = 0$ and hence $z_{f_n} = 0$ (for sufficiently large n).*

For otherwise, say $f_n(0) \neq 0$. Then by (34), 0 is a pole of $\widehat{g}_n(\zeta - w_n/\eta_n)$ with multiplicity at least m, and hence is a pole of $\widehat{g}(\zeta - \alpha)$ with multiplicity at least m. This shows that 0 is not a zero of $G(\zeta) = \zeta^m \widehat{g}(\zeta - \alpha)$, which is a contradiction.

By Claims 12 and 13, we see that if $G(0) = 0$, then $G(\zeta) \neq 0$ for $\zeta \in \mathbb{C}^*$. Hence by Claim 11, G has at most one zero on $\mathbb{C}$.

Claim 14 *If 0 is a zero of G, then the multiplicity is at least $m+k+1$ for $k > 1$ and $2m+3$ for $k = 1$.*

In fact, if 0 is a zero of G, then Claim 13 shows that 0 is a common zero of f_n and h. Let m_{f_n} be the multiplicity of 0 as a zero of f_n. Then by the assumption, $m_{f_n} \geq m+k+1$ for $k > 1$ and $m_{f_n} \geq 2m+3$ for $k = 1$. It follows from (34) that $-w_n/\eta_n$ is a zero of $\widehat{g}_n$ with multiplicity $m_{f_n} - m$, so that $-\alpha$ is a zero of $\widehat{g}$ with multiplicity at least $k+1$ for $k > 1$ and $m+3$ for $k = 1$. The assertion then follows.

Now by (35), we have

$$\eta_n^{-m}[f_n^{(k)}(\eta_n \zeta) - h(\eta_n \zeta)] = G_n^{(k)}(\zeta) - \zeta^m \phi(\eta_n \zeta) \to G^{(k)}(\zeta) - \zeta^m \tag{40}$$

locally uniformly on $\mathbb{C}^* \setminus G^{-1}(\infty)$, or on $\mathbb{C} \setminus G^{-1}(\infty)$ if $\widehat{g}(-\alpha) \neq \infty$.

Claim 15 *$G^{(k)}(\zeta) \not\equiv \zeta^m$.*

For otherwise, $G(\zeta) = c\zeta^{m+k} + P(\zeta)$ for some constant $c \neq 0$ and polynomial P of degree at most $k-1$. Thus, G has at least one zero in $\mathbb{C}$. Let ζ_0 be a zero of G, then by the fact that ζ_0 has multiplicity $> k$, we get $G^{(k)}(\zeta_0) = 0$ and hence $\zeta_0 = 0$ by $G^{(k)}(\zeta) \equiv \zeta^m$. It follows that $G(\zeta) = c'\zeta^l$ for some $c' \in \mathbb{C}$ and $l \in \mathbb{N}$. Thus, $c\zeta^{m+k} + P(\zeta) = c'\zeta^l$. By comparing the degrees and coefficients, we see that $P \equiv 0$. Thus, $G(\zeta) = c\zeta^{m+k}$. This contradicts Claim 14 which says that the multiplicity is at least $m+k+1$ if 0 is a zero of G.

Claim 16 *$G^{(k)}(\zeta) - \zeta^m$ has at most one zero on $\mathbb{C}^*$, or on $\mathbb{C}$ if $\widehat{g}(-\alpha) \neq \infty$.*

By Claim 15, this claim can be seen by applying Hurwitz's theorem to (40) with the condition $f_n^{(k)}(z) \neq h(z)$ for $z \neq z_{f_n}$.

By Claims 10, 11 and 16, it follows from Lemma 6 that G is a non-constant rational function.

Claim 17 *If $G^{(k)}(0)=0$, then $G\neq 0$ on $\mathbb{C}^*$.*

Suppose that $G(\zeta_0)=0$ for some $\zeta_0\neq 0$. Then G is holomorphic at 0. Since we have proved before Claim 14 that G has at most one zero on $\mathbb{C}$, we get $G(0)\neq 0$. Thus by $G(0)\neq 0,\infty$, it follows from $G(\zeta)=\zeta^m\widehat{g}(\zeta-\alpha)$ that $\widehat{g}(-\alpha)=\infty$ and that 0 is a pole of $\widehat{g}(\zeta-\alpha)$ with exact multiplicity m. Thus by (34), there exists a positive number δ such that $f_n(\eta_n\zeta)\neq\infty$ for $|\zeta|\leq\delta$ (and n large enough), and hence $G_n(\zeta)$ is holomorphic on $|\zeta|\leq\delta$. Thus, on $|\zeta|\leq\delta$, the convergence of (40) is uniformly, and then Hurwitz's theorem can be applied to (40). Since $G^{(k)}(0)=0^m$ and $G^{(k)}(\zeta)\not\equiv\zeta^m$ by Claim 15, there exist points $\zeta_n\to 0$ such that $G_n^{(k)}(\zeta_n)=\zeta_n^m\phi(\eta_n\zeta_n)$ so that $f_n^{(k)}(\eta_n\zeta_n)=h(\eta_n\zeta_n)$. Hence, by $f_n^{(k)}(z)\neq h(z)$ for $z\neq z_{f_n}$, we get $z_{f_n}=\eta_n\zeta_n$. On the other hand, since $G(\zeta_0)=0$, applying Hurwitz's theorem to (35) yields that there exist points $\zeta_n'\to\zeta_0$ such that $G_n(\zeta_n')=0$ so that $f_n(\eta_n\zeta_n')=0$. Thus by $f_n(z)\neq 0$ for $z\neq z_{f_n}$, we get $z_{f_n}=\eta_n\zeta_n'$. Hence $\eta_n\zeta_n=\eta_n\zeta_n'(=z_{f_n})$ and then $\zeta_n'=\zeta_n$. It follows that $\zeta_0=0$, which is a contradiction.

Claim 18 *If $G^{(k)}(0)=0$, then $G^{(k)}(\zeta)-\zeta^m\neq 0$ on $\mathbb{C}^*$.*

Suppose that $G^{(k)}(\zeta_0)=\zeta_0^m$ for some $\zeta_0\neq 0$. Then G is holomorphic at 0. If $G(0)=0$, then by Claim 14, $\widehat{g}(-\alpha)=0$, and hence by Claim 16, $G^{(k)}(\zeta)-\zeta^m$ has at most one zero on $\mathbb{C}$. This contradicts that $G^{(k)}(0)=0^m$ and $G^{(k)}(\zeta_0)=\zeta_0^m$. Thus, $G(0)\neq 0$. Now by an argument similar to that in the proof of Claim 17, there exists $\delta>0$ such that on $|\zeta|\leq\delta$, the convergence of (40) is uniformly, and then Hurwitz's theorem can be applied to (40). It follows from $G^{(k)}(0)=0^m$ that there exist points $\zeta_n\to 0$ such that $G_n^{(k)}(\zeta_n)=\zeta_n^m\phi(\eta_n\zeta_n)$ so that $f_n^{(k)}(\eta_n\zeta_n)=h(\eta_n\zeta_n)$. Thus by $f_n^{(k)}(z)=h(z)$ for $z\neq z_{f_n}$, we get $z_{f_n}=\eta_n\zeta_n$. On the other hand, since $G^{(k)}(\zeta_0)=\zeta_0^m$ with $\zeta_0\neq 0$, by applying Hurwitz's theorem to (40), there exist points $\zeta_n'\to\zeta_0$ such that $G_n^{(k)}(\zeta_n')=(\zeta_n')^m\phi(\eta_n\zeta_n')$ so that $f_n^{(k)}(\eta_n\zeta_n')=h(\eta_n\zeta_n')$. Again by $f_n^{(k)}(z)=h(z)$ for $z\neq z_{f_n}$, we get $z_{f_n}=\eta_n\zeta_n'$. Thus $\eta_n\zeta_n=\eta_n\zeta_n'(=z_{f_n})$ and then $\zeta_n'=\zeta_n$. It follows that $\zeta_0=0$, which is a contradiction.

Thus by Claims 16 and 18, $G^{(k)}(\zeta)-\zeta^m$ has at most one zero on $\mathbb{C}$.

Claim 19 *At least one of G and $G^{(k)}(\zeta)-\zeta^m$ has no zero on $\mathbb{C}^*$.*

Suppose that $G(\zeta_0^{(1)})=0$ and $G^{(k)}(\zeta_0^{(2)})=(\zeta_0^{(2)})^m$ for some $\zeta_0^{(1)},\zeta_0^{(2)}\in\mathbb{C}^*$. By Claim 12, $\zeta_0^{(1)}$ is a zero of G with multiplicity $\geq k+2$, and hence $G^{(k)}(\zeta_0^{(1)})=0$. It follows that $\zeta_0^{(1)}\neq\zeta_0^{(2)}$. Applying Hurwitz's theorem to (35) shows that there exist points $\zeta_n^{(1)}\to\zeta_0^{(1)}$ such that $G_n(\zeta_n^{(1)})=0$. Thus $f_n(\eta_n\zeta_n^{(1)})=0$, and hence by $f_n(z)\neq 0$ for $z\neq z_{f_n}$, we get $z_{f_n}=\eta_n\zeta_n^{(1)}$. Applying Hurwitz's theorem to (40) shows that there exist points $\zeta_n^{(2)}\to\zeta_0^{(2)}$ such that $f_n^{(k)}(\eta_n\zeta_n^{(2)})=h(\eta_n\zeta_n^{(2)})$, and hence by $f_n^{(k)}(z)\neq h(z)$ for $z\neq z_{f_n}$, we get $z_{f_n}=\eta_n\zeta_n^{(2)}$. Thus $\zeta_n^{(1)}=\zeta_n^{(2)}$ and hence $\zeta_0^{(1)}=\zeta_0^{(2)}$. This contradicts that $\zeta_0^{(1)}\neq\zeta_0^{(2)}$.

Now, according to Claim 19, we are in one of the following three cases. Note that we have proved before Claim 17 that G is a non-constant rational function.

Case 7 Neither of G and $G^{(k)}(\zeta) - \zeta^m$ has a zero on $\mathbb{C}^*$.

Then by Lemma 11, G has at least one zero on $\mathbb{C}$, and hence $G(0) = 0$. It now follows from Claim 14 and Lemma 12 that Case A cannot occur.

Case 8 $G(\zeta_0) = 0$ for some $\zeta_0 \in \mathbb{C}^*$ and $G^{(k)}(\zeta) - \zeta^m$ has no zero on $\mathbb{C}^*$.

Then $G(\zeta) \neq 0$ for $\zeta \in \mathbb{C} \setminus \{\zeta_0\}$, since we have proved before Claim 14 that G has at most one zero on $\mathbb{C}$. Also, by Claim 17, we have $G^{(k)}(0) \neq 0$, and hence $G^{(k)}(\zeta) \neq \zeta^m$ for $\zeta \in \mathbb{C}$. It now follows from Claim 12 and Lemma 10 that Case B cannot occur.

Case 9 $G^{(k)}(\zeta_0) = \zeta_0^m$ for some $\zeta_0 \in \mathbb{C}^*$ and $G(\zeta)$ has no zero on $\mathbb{C}^*$.

Then $G^{(k)}(\zeta) \neq \zeta^m$ for $\zeta \in \mathbb{C} \setminus \{\zeta_0\}$, since we have proved before Claim 19 that $G^{(k)}(\zeta) - \zeta^m$ has at most one zero on $\mathbb{C}$. Hence by Lemma 11, G has at least one zero on $\mathbb{C}$, so that we must have $G(0) = 0$. It now follows from Claim 14 that $G^{(k)}(0) = 0$, which contradicts that $G^{(k)}(\zeta) \neq \zeta^m$ for $\zeta \in \mathbb{C} \setminus \{\zeta_0\}$.

The proof of Theorem 4 is completed.

4 Proof of Theorem 3

From the proof of Theorem 4, we have the following result, which shows that the multiplicity $k+3$ in Theorem 4 can be replaced by $k+2$ when the set E_f is independent of f. Note that the assumption that multiplicity $\geq k+3$ in Theorem 4 is only used to rule out the Case B within the proof of Theorem 4.

Theorem 13 *Let $k \in \mathbb{N}$ and $h(\not\equiv 0)$ be a function holomorphic on D. Let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least $k+2$. Suppose that there exists a point $z_0 \in D$ such that on $D \setminus \{z_0\}$, $f \neq 0$ and $f^{(k)} \neq h$ for all functions $f \in \mathcal{F}$. Suppose additionally that if z_0 is a common zero of $f \in \mathcal{F}$ and h, the multiplicities m_f for f and m_h for h satisfy $m_f \geq m_h + k + 1$ for $k > 1$ and $m_f \geq 2m_h + 3$ for $k = 1$. Then $\mathcal{F}$ is normal in D.*

Proof The proof is similar to, and more simple than, the proof of Theorem 4, since in this case, we only have to consider the case that $z_{f_n} = 0$ for all n in the proof of Theorem 4. In fact, since $z_{f_n} = 0$, we see from Claim 12 that G has no zeros on $\mathbb{C}^*$, and a similar argument by applying Hurwitz's theorem to (40) can be used to show that $G^{(k)}(\zeta) - \zeta^m$ has no zeros on $\mathbb{C}^*$. So only the Case A appears and is required to be ruled out. □

Proof of Theorem 3 Let $z_0 \in D$ and a sequence $\{f_n\} \subset \mathcal{F}$. We have to prove that $\{f_n\}$ is normal at z_0.

Suppose first that there exists a neighborhood U_0 of z_0 such that on $U_0 \setminus \{z_0\}$, $f_1 \neq 0$ and $f_1^{(k)} \neq h$. Then by the condition, on $U_0 \setminus \{z_0\}$, $f_n \neq 0$ and $f_n^{(k)} \neq h$ for all functions f_n. Hence, by Theorem 13, $\{f_n\}$ is normal in U_0 and hence normal at z_0.

The other case is that there exists a sequence $z_n \to z_0$, $z_n \neq z_0$ such that either $f_1(z_n) = 0$ or $f_1^{(k)}(z_n) = h(z_n)$. It follows that either $f_1(z) \equiv 0$ or $f_1^{(k)}(z) \equiv h(z)$. Thus by the condition, either $f_n(z) \equiv 0$ or $f_n^{(k)}(z) \equiv h(z)$ for all n.

It is obviously that $\{f_n\}$ is normal if $f_n(z) \equiv 0$. Thus, we consider the case that $f_n^{(k)}(z) \equiv h(z)$. If $\{f_n\}$ is not normal at z_0, then by Lemma 5, there exists points $z_n \rightarrow z_0$, positive numbers $\rho_n \rightarrow 0$ and functions $f_n \in \{f_n\}$ such that $g_n(\zeta) = f_n(z_n + \rho_n\zeta) \rightarrow g(\zeta)$ locally uniformly on $\mathbb{C}$, where g is a non-constant meromorphic function, all of whose zeros have multiplicity at least $k+2$. Since $g_n^{(k)}(\zeta) = \rho_n^k f_n^{(k)}(z_n + \rho_n\zeta) = \rho_n^k h(z_n + \rho_n\zeta) \rightarrow 0$, we get $g^{(k)}(\zeta) \equiv 0$. Since all zeros of g have multiplicity at least $k+2$, it follows that g is a constant, which is a contradiction.

Thus, $\{f_n\}$ is normal at z_0. Theorem 3 is proved.

Acknowledgments The author would like to thank the referee for his valuable suggestions and comments, which helps a lot in improving the paper.

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